PHYS 206
Homework Assignment 1

Jody Steele
07367455

September 24, 2026

Problem 1

Prove Euler's identity,

$$e^{ix} = \cos x + i \sin x,$$

by showing that the Taylor series of both sides of the equation are the same to all orders in x.

Proof. The Taylor series for e^{ix} is

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

We know that cos and sin are even and odd functions respectively, so lets break the sum for e^{ix} up into it's even and odd constituents. We use the fact that for an arbitrary value n, $2n$ gives us only even terms and $2n+1$ gives us only odd terms. We can then rewrite the sum as follows:

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}$$

Additionally, because $i^2 = -1$, i^{2n} can be replaced with $(-1)^n$. We can also do this with the second term $i^{2n+1} = i(i^{2n}) = i(-1)^n$, moving the remaining i outside of the sum:

$$e^{ix} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

From class we have the Taylor series for $\cos x$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

and $\sin x$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Replacing the LHS and RHS from the original equation with their respective sums we get

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

We see that the LHS=RHS, thus $e^{ix} = \cos x + i \sin x$ $\square$

Problem 2

Consider the motion of a particle travelling along a helix at constant speed. Suppose the particle's position vector as a function of time is given by:

$$\vec{r}(t) = \rho\cos\omega t\ \hat{x} + \rho\sin\omega t\ \hat{y} + vt\ \hat{z},$$

where ρ, ω, v are positive constants. At every point along the particle's path we can define three unit vectors: $\hat{T}$, $\hat{N}$, and $\hat{B}$. The first is the tangent, meaning it is the unit vector pointing in the direction of the velocity, i.e. $\hat{T} = \dot{\vec{r}}/|\dot{\vec{r}}|$. The second is called the normal, which is the unit vector pointing in the direction of the acceleration, i.e. $\hat{N} = \ddot{\vec{r}}/|\ddot{\vec{r}}|$. The third is called the binormal, and is simply $\hat{B} = \hat{T} \times \hat{N}$. Calculate $\hat{T}$, $\hat{N}$, and $\hat{B}$, and calculate the constants κ and τ (in terms of ρ, ω, and v) in the following relations:

$$\dot{\hat{T}} = \kappa\hat{N}, \dot{\hat{N}} = -\kappa\hat{T} + \tau\hat{B}, \dot{\hat{B}} = -\tau\hat{N}.$$

We need to find $\dot{\vec{r}}$ and $\ddot{\vec{r}}$, so lets do those first

$$\dot{\vec{r}} = -\rho\omega\sin\omega t\ \hat{x} + \rho\omega\cos\omega t\ \hat{y} + v\ \hat{z}$$

$$\ddot{\vec{r}} = -\rho\omega^2\cos\omega t\ \hat{x} - \rho\omega^2\sin\omega t\ \hat{y}$$

We'll also need their magnitudes:

$$\begin{aligned}
|\dot{\vec{r}}| &= \sqrt{(-\rho\omega\sin\omega t)^2 + (\rho\omega\cos\omega t)^2 + (v)^2} \\
&= \sqrt{(\rho\omega)^2\sin^2\omega t + (\rho\omega)^2\cos^2\omega t + v^2} \\
&= \sqrt{(\rho\omega)^2(\sin^2\omega t + \cos^2\omega t) + v^2} \\
&= \sqrt{(\rho\omega)^2 + v^2}
\end{aligned}$$

$$\begin{aligned}
|\ddot{\vec{r}}| &= \sqrt{(-\rho\omega^2\cos\omega t)^2 + (-\rho\omega^2\sin\omega t)^2} \\
&= \sqrt{(-\rho\omega^2)^2\cos^2\omega t + (-\rho\omega^2)^2\sin^2\omega t} \\
&= \sqrt{(\rho\omega^2)^2(\cos^2\omega t + \sin^2\omega t)} \\
&= \sqrt{(\rho\omega^2)^2} \\
&= \rho\omega^2
\end{aligned}$$

Now we can calculate $\hat{T}$ $\hat{N}$, and $\hat{B}$

$$\begin{aligned}
\hat{T} &= \dot{\vec{r}}/|\dot{\vec{r}}| \\
&= \frac{-\rho\omega\sin\omega t\ \hat{x} + \rho\omega\cos\omega t\ \hat{y} + v\ \hat{z}}{\sqrt{(\rho\omega)^2 + v^2}} \\
&= \frac{-\rho\omega\sin\omega t}{\sqrt{(\rho\omega)^2 + v^2}}\hat{x} + \frac{\rho\omega\cos\omega t}{\sqrt{(\rho\omega)^2 + v^2}}\hat{y} + \frac{v}{\sqrt{(\rho\omega)^2 + v^2}}\hat{z}
\end{aligned}$$

$$\begin{aligned}
\hat{N} &= \ddot{\vec{r}}/|\ddot{\vec{r}}| \\
&= \frac{-\rho\omega^2\cos\omega t\ \hat{x} - \rho\omega^2\sin\omega t\ \hat{y}}{\rho\omega^2} \\
&= -\cos\omega t\ \hat{x} - \sin\omega t\ \hat{y}
\end{aligned}$$

$$
\begin{aligned}
\hat{B} &= \hat{T} \times \hat{N} \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{-\rho\omega \sin \omega t}{\sqrt{(\rho\omega)^2+v^2}} & \frac{\rho\omega \cos \omega t}{\sqrt{(\rho\omega)^2+v^2}} & \frac{v}{\sqrt{(\rho\omega)^2+v^2}} \\ -\cos \omega t & -\sin \omega t & 0 \end{vmatrix} \\
&= \frac{v \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{x} - \frac{v \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{y} + \frac{\rho\omega \sin^2 \omega t + \rho\omega \cos^2 \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{z} \\
&= \frac{v \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{x} - \frac{v \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{y} + \frac{\rho\omega}{\sqrt{(\rho\omega)^2 + v^2}} \hat{z} \\
&= \frac{v \sin \omega t \ \hat{x} - v \cos \omega t \ \hat{y} + \rho\omega \ \hat{z}}{\sqrt{(\rho\omega)^2 + v^2}}
\end{aligned}
$$

Given $\dot{\hat{T}} = \kappa \hat{N}$, $\hat{N} = \ddot{\vec{r}}/|\ddot{\vec{r}}|$, $\hat{T} = \dot{\vec{r}}/|\dot{\vec{r}}|$, and from above we can see that $|\dot{\vec{r}}|$ is not dependant on t, we can see that $\dot{\hat{T}} = \ddot{\vec{r}}/|\dot{\vec{r}}|$, and by re-arranging we get:

$$
\begin{aligned}
\dot{\hat{T}} &= \kappa \hat{N} \\
\frac{\ddot{\vec{r}}}{|\dot{\vec{r}}|} &= \kappa \frac{\ddot{\vec{r}}}{|\ddot{\vec{r}}|} \\
\kappa &= \frac{|\ddot{\vec{r}}|}{|\dot{\vec{r}}|} \\
\kappa &= \frac{\rho\omega^2}{\sqrt{(\rho\omega)^2 + v^2}}
\end{aligned}
$$

To find τ we'll first need to calculate $\dot{\hat{B}}$:

$$
\begin{aligned}
\dot{\hat{B}} &= \frac{d}{dt} \left(\frac{v \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{x} - \frac{v \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{y} + \frac{\rho\omega}{\sqrt{(\rho\omega)^2 + v^2}} \hat{z} \right) \\
&= \frac{v\omega \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{x} + \frac{v\omega \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{y}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\hat{B}} &= -\tau \hat{N} \\
\frac{v\omega \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{x} + \frac{v\omega \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \hat{y} &= -\tau(-\cos \omega t \ \hat{x} - \sin \omega t \ \hat{y}) \\
\frac{v\omega}{\sqrt{(\rho\omega)^2 + v^2}} (\cos \omega t \hat{x} + \sin \omega t \hat{y}) &= \tau(\cos \omega t \ \hat{x} + \sin \omega t \ \hat{y}) \\
\tau &= \frac{v\omega}{\sqrt{(\rho\omega)^2 + v^2}}
\end{aligned}
$$

Finally, we can use the $\dot{\hat{N}} = -\kappa \hat{T} + \tau \hat{B}$ equation to confirm κ and τ.

$$
\begin{aligned}
\dot{\hat{N}} &= \frac{d}{dt} (-\cos \omega t \ \hat{x} - \sin \omega t \ \hat{y}) \\
&= \omega \sin \omega t \ \hat{x} - \omega \cos \omega t \ \hat{y}
\end{aligned}
$$

Tackling it piecewise

$$
\begin{aligned}
\dot{\hat{N}}_x &= -\kappa \hat{T}_x + \tau \hat{B}_x \\
\omega \sin \omega t \ \hat{x} &= \left(\frac{-\rho\omega^2}{\sqrt{(\rho\omega)^2 + v^2}} \right) \left(\frac{-\rho\omega \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \right) + \left(\frac{v\omega}{\sqrt{(\rho\omega)^2 + v^2}} \right) \left(\frac{v \sin \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \right) \hat{x} \\
&= \frac{-(\rho\omega^2)(-\rho\omega \sin \omega t) + (v\omega)(v \sin \omega t)}{(\rho\omega)^2 + v^2} \hat{x} \\
&= \frac{\rho^2\omega^3 \sin \omega t + v^2\omega \sin \omega t}{\rho^2\omega^2 + v^2} \hat{x} \\
&= \frac{(\omega \sin \omega t)(\rho^2\omega^2 + v^2)}{\rho^2\omega^2 + v^2} \hat{x} \\
\omega \sin \omega t \ \hat{x} &= \omega \sin \omega t \ \hat{x}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\hat{N}}_y &= -\kappa \hat{T}_y + \tau \hat{B}_y \\
-\omega \cos \omega t \ \hat{y} &= \left(\frac{-\rho\omega^2}{\sqrt{(\rho\omega)^2 + v^2}} \right) \left(\frac{\rho\omega \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \right) + \left(\frac{v\omega}{\sqrt{(\rho\omega)^2 + v^2}} \right) \left(\frac{-v \cos \omega t}{\sqrt{(\rho\omega)^2 + v^2}} \right) \hat{y} \\
&= \frac{-(\rho\omega^2)(\rho\omega \cos \omega t) + (v\omega)(-v \cos \omega t)}{(\rho\omega)^2 + v^2} \hat{y} \\
&= \frac{-\rho^2\omega^3 \cos \omega t - v^2\omega \cos \omega t}{\rho^2\omega^2 + v^2} \hat{y} \\
&= \frac{(-\omega \cos \omega t)(\rho^2\omega^2 + v^2)}{\rho^2\omega^2 + v^2} \hat{y} \\
-\omega \cos \omega t \ \hat{y} &= -\omega \cos \omega t \ \hat{y}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\hat{N}}_z &= -\kappa \hat{T}_z + \tau \hat{B}_z \\
0 \ \hat{z} &= \left(\frac{-\rho\omega^2}{\sqrt{(\rho\omega)^2 + v^2}} \right) \left(\frac{v}{\sqrt{(\rho\omega)^2 + v^2}} \right) + \left(\frac{v\omega}{\sqrt{(\rho\omega)^2 + v^2}} \right) \left(\frac{\rho\omega}{\sqrt{(\rho\omega)^2 + v^2}} \right) \hat{z} \\
&= \frac{-(\rho\omega^2)(v) + (v\omega)(\rho\omega)}{(\rho\omega)^2 + v^2} \hat{z} \\
&= \frac{-\rho\omega^2 v + \rho\omega^2 v}{(\rho\omega)^2 + v^2} \hat{z} \\
0 \ \hat{z} &= 0 \ \hat{z}
\end{aligned}
$$

Problem 3

In lecture we started with the differential equations $\ddot{x} = 0$ and $\ddot{x} + \omega^2 x = 0$, and integrated twice to find $x(t)$. Suppose this time we had a physical system of some kind whose differential equation is given by

$$m\ddot{x} = -u\dot{m},$$

where $u > 0$ is a constant, and x and m are both functions of time. This is called the ideal rocket equation. x is the position of the rocket, m is its mass.

Do the first integration without making any assumptions about the functional form of m. After the first integration, assume $m(t) = m_0(1 - t/T)$, and then perform the second integration. Choose your constants of integration so that $x(0) = 0$, and $\dot{x}(0) = 0$, i.e. the rocket is initially at the origin and not moving.

Lets re-write the equation so we can abuse some derivative notation!

$$\begin{aligned}
m\frac{d}{dt}\dot{x} &= -u\frac{d}{dt}m \\
m\frac{dv}{dt} &= -u\frac{dm}{dt} \\
\frac{dv}{dt} &= \frac{-u}{m}\frac{dm}{dt} \\
dv &= \frac{-u}{m}\frac{dm}{dt}dt \\
dv &= -u\frac{1}{m}dm
\end{aligned}$$

And then integrate both sides[1]

$$\begin{aligned}
\int dv &= \int -u\frac{1}{m}dm \\
v &= -u\ln m + C
\end{aligned}$$

Subbing in our assumed equation for m

$$\begin{aligned}
v &= -u\ln\left(m_0\left(1 - \frac{t}{T}\right)\right) + C \\
v &= -u\left(\ln(m_0) + \ln\left(1 - \frac{t}{T}\right)\right) + C
\end{aligned}$$

With $\dot{x}(0) = 0$ we can determine the value for C now

$$\begin{aligned}
0 &= -u\left(\ln(m_0) + \ln\left(1 - \frac{0}{T}\right)\right) + C \\
0 &= -u(\ln(m_0) + \ln(1)) + C \\
0 &= -u\ln(m_0) + C \\
C &= u\ln(m_0)
\end{aligned}$$

[1] "log" has always meant $\log_{10}$ for me growing up, so I'm going to use ln

Plugging this back into the original equation and simplifying we get

$$
\begin{aligned}
v &= -u\left(\ln(m_0) + \ln\left(1 - \frac{t}{T}\right)\right) + u\ln(m_0) \\
v &= -u\ln(m_0) - u\ln\left(1 - \frac{t}{T}\right) + u\ln(m_0) \\
v &= -u\ln\left(1 - \frac{t}{T}\right)
\end{aligned}
$$

Once again, let's rewrite this to abuse the notation

$$
\begin{aligned}
\frac{dx}{dt} &= -u\ln\left(1 - \frac{t}{T}\right) \\
dx &= -u\ln\left(1 - \frac{t}{T}\right)\ dt
\end{aligned}
$$

We'll do u-substitution, but with z because we have a u already. Let $z = 1 - \frac{t}{T}$ then $dz = \frac{-1}{T}dt$ and $dt = -T\ dz$. Subbing this in

$$
\begin{aligned}
dx &= -u\ln(z)(-T\ dz) \\
dx &= uT\ln(z)\ dz \\
\int dx &= uT\int \ln z\ dz
\end{aligned}
$$

The left side is trivial, but the right side requires integration by parts. We'll use $w = \ln z$ and $dv = dz$ which gives us $dw = 1/z\ dz$ and $v = z$.

$$
\begin{aligned}
\int dx &= uT\int \ln z\ dz \\
\int dx &= uT\left(\ln(z)(z) - \int (z)\frac{1}{z}dz\right) \\
\int dx &= uT\left(\ln(z)(z) - \int dz\right) \\
x &= uT\left(\ln(z)(z) - z + C\right)
\end{aligned}
$$

Subbing z back in

$$
\begin{aligned}
x &= uT\left(\ln(z)(z) - z + C\right) \\
x &= uT\left(\ln\left(1 - \frac{t}{T}\right)\left(1 - \frac{t}{T}\right) - \left(1 - \frac{t}{T}\right) + C\right)
\end{aligned}
$$

With $x(0) = 0$ we've got enough to find the second constant

$$
\begin{aligned}
0 &= uT\left(\ln\left(1 - \frac{0}{T}\right)\left(1 - \frac{0}{T}\right) - \left(1 - \frac{0}{T}\right) + C\right) \\
0 &= uT\left(\ln(1)(1) - (1) + C\right) \\
0 &= -1 + C \\
C &= 1
\end{aligned}
$$

Plugging in, and simplifying

$$x = uT\left(\ln\left(1-\frac{t}{T}\right)\left(1-\frac{t}{T}\right)-\left(1-\frac{t}{T}\right)+1\right)$$
$$x = uT\left(\ln\left(1-\frac{t}{T}\right)\left(1-\frac{t}{T}\right)+\frac{t}{T}\right)$$
$$x = u\left(\ln\left(1-\frac{t}{T}\right)(T-t)+t\right)$$